Theory and Applications of Categories, Vol. 34, No. 46, 2019, pp. 1526–1551.

CATEGORIES OF WEAK FRACTIONS

PIERRE-ALAIN JACQMIN

Abstract. Given a set Σ of morphisms in a category $\mathcal{C}$, we construct a functor $F_{1/\Sigma}\colon \mathcal{C} \to \mathcal{C}[1/\Sigma]$ which sends elements of Σ to split monomorphisms. Moreover, we prove that $F_{1/\Sigma}$ is weakly universal with that property when considered in the world of locally posetal 2-categories. Besides, we also use locally posetal 2-categories in order to construct weak left adjoints to those functors for which any object in the codomain admits a weak reflection. We then apply these two results in order to restate the Injective Subcategory Problem for Σ into the existence of some kind of weak right adjoint for $F_{1/\Sigma}$.

1. Introduction

While a universal property requires the existence of a unique morphism satisfying a given property, a *weak* universal property merely requires its existence (but not its uniqueness), see [5, 9, 19]. For instance, given a functor $G\colon \mathcal{D} \to \mathcal{C}$, a reflection (respectively a weak reflection) for an object C in $\mathcal{C}$ along G consists of an object D in $\mathcal{D}$ together with a morphism $n\colon C \to G(D)$ such that for any morphism $m\colon C \to G(D')$, there exists a unique (respectively, there exists a) morphism $d\colon D \to D'$ such that $G(d) \circ n = m$.

$$\begin{tikzcd} C \arrow[r, "n"] \arrow[dr, "m"'] & G(D) \arrow[d, dotted, "G(d)"] \\ & G(D') \end{tikzcd}$$

Given such a reflection along G for each object C in $\mathcal{C}$, it is well known that one can construct a left adjoint $F\colon \mathcal{C} \to \mathcal{D}$ to G, the value of F on arrows being determined by these universal properties. If instead, each object C of $\mathcal{C}$ is only endowed with a weak reflection along G, such a left adjoint F can no longer be constructed; the reason being that the image by F of an arrow is not uniquely determined and nothing ensures that a functorial choice can be made. To get round this, *weak left adjoints* have been considered in the literature (see [17, 19]). A closely related, but more general, approach than in [17] is considered here, using categories enriched in the category Pos of posets. Pos-categories

Support from NSERC is gratefully acknowledged.

Received by the editors 2019-08-30 and, in final form, 2019-12-16.

Transmitted by Jiri Rosicky. Published on 2019-12-18.

2010 Mathematics Subject Classification: 55P60, 18A40, 18G05 (primary); 18A20, 18D05, 18B35, 18A10 (secondary).

Key words and phrases: split monomorphism, locally posetal 2-category, category of fractions, weak reflection, weak adjoint, injective subcategory problem.

can be seen as ordinary categories coherently equipped with a poset structure on each hom-set. We regard each category $\mathcal{C}$ as a Pos-category $\mathcal{P}_0(\mathcal{C})$ having the same objects as $\mathcal{C}$ but whose hom-set $\mathcal{P}_0(\mathcal{C})(A,B)$ is the set of non-empty subsets of $\mathcal{C}(A,B)$, ordered by inclusion. A functor $G\colon \mathcal{D} \to \mathcal{C}$ between ordinary categories induces a Pos-functor $\mathcal{P}_0(G)\colon \mathcal{P}_0(\mathcal{D}) \to \mathcal{P}_0(\mathcal{C})$. If each object of $\mathcal{C}$ admits a weak reflection along G, one can now construct a weak left adjoint $F\colon \mathcal{P}_0(\mathcal{C}) \rightharpoonup \mathcal{P}_0(\mathcal{D})$ in the sense of Definition 3.3. The $\rightharpoonup$ sign indicates that F is merely a lax-Pos-functor, i.e., identities and composition are preserved laxly: $1_{F(A)} \leqslant F(1_A)$ and $F(g) \circ F(f) \leqslant F(g \circ f)$. The construction of F is now based on the fact that, for a non-empty set of arrows $C \to C'$, one can associate the (non-empty) set of *all* arrows $F(C) \to F(C')$ satisfying the obvious commutativity condition.

Given a class Σ of morphisms in a category $\mathcal{C}$, the *category of fractions* of $\mathcal{C}$ with respect to Σ is, if it exists, the universal functor $F_{\Sigma^{-1}}\colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ which sends elements of Σ to isomorphisms [11]. When Σ is a set, the category of fractions always exists and is constructed by formally adding inverses to elements of Σ. The main objective of this paper is to treat the weak version of it, namely to find a weakly universal functor which sends elements of Σ to split monomorphisms. In order to avoid size problems, we will assume that Σ is a set of morphisms of $\mathcal{C}$. Then, for each element $s\colon A \to B$ of Σ, we formally add a morphism $\overline{s}\colon B \to A$ such that $\overline{s} \circ s = 1_A$ to form the category $\mathcal{C}[1/\Sigma]$. This category can thus be thought of as adding left inverses to elements of Σ. It comes equipped with a functor $F_{1/\Sigma}\colon \mathcal{C} \to \mathcal{C}[1/\Sigma]$ which sends elements of Σ to split monomorphisms. Then, given a functor $G\colon \mathcal{C} \to \mathcal{D}$ which also sends elements of Σ to split monomorphisms, there exists a Pos-functor $H\colon \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \to \mathcal{P}_0(\mathcal{D})$ such that $H \circ \mathcal{P}_0(F_{1/\Sigma}) = \mathcal{P}_0(G)$.

$$\begin{tikzcd}
\mathcal{P}_0(\mathcal{C}) \arrow[r, "\mathcal{P}_0(F_{1/\Sigma})"] \arrow[dr, "\mathcal{P}_0(G)"'] & \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \arrow[d, dotted, "\exists\, H"] \\
& \mathcal{P}_0(\mathcal{D})
\end{tikzcd}$$

One also has such a weak universal property in the case where G is a lax-Pos-functor $\mathcal{P}_0(\mathcal{C}) \rightharpoonup \mathbb{D}$ to a Pos-category $\mathbb{D}$. In that case, one needs to require that $\mathbb{D}$ is *almost complete*, i.e., that hom-sets in $\mathbb{D}$ have non-empty suprema and these are preserved by left and right compositions. One also needs G to be *almost complete*, i.e., that the action of G on arrows preserves non-empty suprema. The assumption that G sends elements of Σ to split monomorphisms is now replaced by the conjunction of:

(A) for all $s\colon A \to B$ in Σ, there exists $d \in \mathbb{D}(G(B), G(A))$ such that $d \circ G(sf) \leqslant G(f)$ for any $f \in \mathcal{C}(C, A)$;

(B) if $G(sf) = G(sf')$ for some $f, f'\colon C \to A$ in $\mathcal{C}$ and $s\colon A \to B$ in Σ, then $G(hf) = G(hf')$ for any $h \in \mathcal{C}(A, D)$.

Condition (A) should be thought of as the existence of a retraction for $G(s)$ while condition (B) mimics the fact that $G(s)$ should be a monomorphism. When these conditions hold, there exists an almost complete lax-Pos-functor $H\colon \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \rightharpoonup \mathbb{D}$ such that $H \circ \mathcal{P}_0(F_{1/\Sigma}) = G$.

$$\begin{tikzcd}
\mathcal{P}_0(\mathcal{C}) \arrow[r, "\mathcal{P}_0(F_{1/\Sigma})"] \arrow[dr, "G"'] & \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \arrow[d, dotted, "\exists\, H"] \\
& \mathbb{D}
\end{tikzcd}$$

This problem of turning elements of Σ into split monomorphisms recently appeared in [15] where, under strong conditions on Σ, a functorial choice of retractions has been constructed and the solution was proved to be (strongly) universal. The analogous problem of turning some morphisms of a Pos-category into left adjoint split monomorphisms has been considered in [21].

Given a class Σ of morphisms in a category $\mathcal{C}$, an object I is said to be *orthogonal to* (respectively, *injective with respect to*) Σ if the presheaf $\mathcal{C}(-,I)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ sends elements of Σ to isomorphisms (respectively to epimorphisms). The class of objects orthogonal to Σ (respectively, injective with respect to Σ) is denoted by $\Sigma^{\perp}$ (respectively by $\Sigma^{\vartriangle}$). The *Orthogonal Subcategory Problem* (OSP) is the problem to know whether $\Sigma^{\perp}$ is a reflective subcategory of $\mathcal{C}$. This problem appeared in [10] and has been deeply studied since then (see [1, 3, 4, 6, 18, 20]). In [11], connections with the category of fractions $\mathcal{C}[\Sigma^{-1}]$ are established. One can show that, if Σ is a set, the full inclusion $\Sigma^{\perp} \hookrightarrow \mathcal{C}$ has a left adjoint whose unit is inverted by $F_{\Sigma^{-1}}$ if and only if $F_{\Sigma^{-1}}\colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ has a fully faithful right adjoint (see Theorem 5.3).

Due to the new technology developed here, one can have a similar reformulation for the weak version of the OSP, i.e., the *Injective Subcategory Problem* (ISP). The ISP asks whether $\Sigma^{\vartriangle}$ is weakly reflective in $\mathcal{C}$ (see [1, 2, 13, 14] for instance). If each object C of $\mathcal{C}$ has a weak reflection $n_C\colon C \to J(D)$ along the full inclusion $J\colon \Sigma^{\vartriangle} \hookrightarrow \mathcal{C}$ for which $n_C \in \Sigma$, one can construct a weak left adjoint $R\colon \mathcal{P}_0(\mathcal{C}) \rightharpoonup \mathcal{P}_0(\Sigma^{\vartriangle})$ to $\mathcal{P}_0(J)$. By the weak universal property of $F_{1/\Sigma}\colon \mathcal{C} \to \mathcal{C}[1/\Sigma]$, this gives rise to a lax-Pos-functor $R'\colon \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \rightharpoonup \mathcal{P}_0(\Sigma^{\vartriangle})$. The composite $\mathcal{P}_0(J) \circ R'$ has then a property close to being a weak right adjoint for $\mathcal{P}_0(F_{1/\Sigma})$, and this property is equivalent to the condition of having weak reflections along J formed by elements of Σ, see Theorem 6.1.

$$\begin{tikzcd}
& \mathcal{P}_0(\Sigma^{\vartriangle}) \arrow[dl, hook', "\mathcal{P}_0(J)"'] & \\
\mathcal{P}_0(\mathcal{C}) \arrow[ur, harpoon, "R"'] \arrow[rr, "\mathcal{P}_0(F_{1/\Sigma})"'] & & \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \arrow[ul, harpoon, "R'"']
\end{tikzcd}$$

The paper is organized as follows. In Section 2, we establish the terminology on Pos-categories needed in the paper. In Section 3, we give equivalent definitions of a weak left adjoint for a Pos-functor. In Section 4, we construct the functor $F_{1/\Sigma}\colon \mathcal{C} \to \mathcal{C}[1/\Sigma]$ and prove its weak universal property. Section 5 is devoted to the reformulation of the OSP using the category of fractions. Finally, in Section 6, we use the results from the previous sections in order to reformulate the ISP in terms of the functor $F_{1/\Sigma}\colon \mathcal{C} \to \mathcal{C}[1/\Sigma]$.

2. Pos-categories

Let Pos be the cartesian closed category of posets and order-preserving maps.

2.1. Definition. *A* Pos-category *is a* Pos*-enriched category. Equivalently, it is a locally posetal 2-category. A* Pos-functor *between* Pos*-categories is a* Pos*-enriched functor, or equivalently, it is a 2-functor between the locally posetal 2-categories.*

2.2. Definition. *Let $\mathbb{C}$ and $\mathbb{D}$ be two* Pos*-categories. A* lax-Pos-functor $F\colon \mathbb{C} \rightharpoonup \mathbb{D}$ *is a lax-2-functor between the locally posetal 2-categories $\mathbb{C}$ and $\mathbb{D}$. Equivalently, it is given by*

- *a function $F\colon \operatorname{ob}(\mathbb{C}) \to \operatorname{ob}(\mathbb{D})$,*
- *for each pair of objects (A, B) in $\mathbb{C}$, an order-preserving function $F\colon \mathbb{C}(A, B) \to \mathbb{D}(F(A), F(B))$*

such that

- *$1_{F(A)} \leqslant F(1_A)$ for each object A of $\mathbb{C}$,*
- *$F(g) \circ F(f) \leqslant F(g \circ f)$ for any pair (f, g) of composable arrows in $\mathbb{C}$.*

2.3. Notation. Each Pos-category $\mathbb{C}$ induces a Pos-functor $\mathbb{C}(-,-)\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Pos}$ which is the classical 'hom-functor'. As usual, the category Pos is seen as a Pos-category with the pointwise order on morphisms and the Pos-category $\mathbb{C}^{\mathrm{op}}$ is obtained by reversing the 1-cells of $\mathbb{C}$ but not the order in the hom-sets.

2.4. Definition. *Let $F, G\colon \mathbb{C} \rightharpoonup \mathbb{D}$ be lax-*Pos*-functors between* Pos*-categories. A* Pos-transformation $\alpha\colon F \rightsquigarrow G$ *consists of, for each object A of $\mathbb{C}$, a morphism $\alpha_A\colon F(A) \to G(A)$. We say that*

- *α is* lax-natural *($\alpha\colon F \rightharpoonup G$) if $G(f) \circ \alpha_A \leqslant \alpha_B \circ F(f)$ for each $f \in \mathbb{C}(A, B)$;*
- *α is* oplax-natural *($\alpha\colon F \rightharpoondown G$) if $G(f) \circ \alpha_A \geqslant \alpha_B \circ F(f)$ for each $f \in \mathbb{C}(A, B)$;*
- *α is* natural *($\alpha\colon F \to G$) if it is both lax-natural and oplax-natural, i.e., if $G(f) \circ \alpha_A = \alpha_B \circ F(f)$ for each $f \in \mathbb{C}(A, B)$.*

We have evident identities and compositions of lax-Pos-functors and of Pos-transformations. Moreover, given two Pos-transformations $\alpha, \beta\colon F \rightsquigarrow G$, we write $\alpha \leqslant \beta$ if $\alpha_A \leqslant \beta_A$ for all objects A of $\mathbb{C}$.

In [12], a poset P is said to be an *almost compete semilattice* if it has all non-empty suprema. In view of this, we define:

2.5. Definition. *A* Pos*-category* $\mathbb{C}$ *is said to be* almost complete *if for all pair of objects* (A, B)*, the poset* $\mathbb{C}(A, B)$ *has all non-empty suprema and if for each morphism* $f\colon A \to B$ *and each object* X *in* $\mathcal{C}$*, the composition maps*

$$f \circ -\colon \mathbb{C}(X, A) \to \mathbb{C}(X, B) \quad \textit{and} \quad - \circ f\colon \mathbb{C}(B, X) \to \mathbb{C}(A, X)$$

preserve non-empty suprema.

*A lax-*Pos*-functor* $F\colon \mathbb{C} \rightharpoonup \mathbb{D}$ *between* Pos*-categories is said to be* almost complete *if, for each pair* (A, B) *of objects of* $\mathbb{C}$*, the map*

$$F\colon \mathbb{C}(A, B) \to \mathbb{D}(F(A), F(B))$$

preserves existing non-empty suprema.

Note that in general, an almost complete Pos-category is not enriched in the category of almost complete semilattices and functions preserving non-empty suprema since, for non-rectangular non-empty sets $S \subseteq \mathbb{C}(A, B) \times \mathbb{C}(B, C)$, we *do not* have

$$\left(\bigvee_{g \in \pi_2(S)} g \right) \circ \left(\bigvee_{f \in \pi_1(S)} f \right) = \bigvee_{(f,g) \in S} (g \circ f)$$

in general.

2.6. Notation. For a set X, we denote by $\mathcal{P}_0(X)$ the set

$$\mathcal{P}_0(X) = \{X' \subseteq X \,|\, X' \neq \varnothing\}.$$

To any category $\mathcal{C}$, we associate an almost complete Pos-category $\mathcal{P}_0(\mathcal{C})$ having the same objects as $\mathcal{C}$ and for any pair of objects (A, B) in $\mathcal{C}$, $\mathcal{P}_0(\mathcal{C})(A, B) = \mathcal{P}_0(\mathcal{C}(A, B))$ ordered by inclusion. Given $X \in \mathcal{P}_0(\mathcal{C}(A, B))$ and $Y \in \mathcal{P}_0(\mathcal{C}(B, C))$, we define

$$Y \circ X = \{g \circ f \,|\, f \in X, g \in Y\}.$$

The identity on A in $\mathcal{P}_0(\mathcal{C})$ is the singleton $\{1_A\}$.

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ between ordinary categories, one constructs an almost complete Pos-functor $\mathcal{P}_0(F)\colon \mathcal{P}_0(\mathcal{C}) \to \mathcal{P}_0(\mathcal{D})$ by $\mathcal{P}_0(F)(A) = F(A)$ for any object A in $\mathcal{C}$ and $\mathcal{P}_0(F)(X) = \{F(f) \,|\, f \in X\}$ for $X \in \mathcal{P}_0(\mathcal{C}(A, B))$. Finally, each natural transformation $\alpha\colon F \to G\colon \mathcal{C} \to \mathcal{D}$ gives rise to a natural Pos-transformation $\mathcal{P}_0(\alpha)\colon \mathcal{P}_0(F) \to \mathcal{P}_0(G)$ simply by setting $\mathcal{P}_0(\alpha)_A = \{\alpha_A\}$ for each object A of $\mathcal{C}$.

By abuse of notation, for a lax-Pos-functor $G\colon \mathcal{P}_0(\mathcal{C}) \rightharpoonup \mathbb{D}$, we are often going to denote $G(\{f\})$ by $G(f)$ for a morphism f in $\mathcal{C}$.

3. Weak left adjoints

Let us now turn our attention to weak adjunctions. The following theorem is the 'weak version' of the well-know fact that an adjunction $F \dashv G$ can be equivalently described via a unit $\eta\colon 1_{\mathcal{C}} \to GF$ and a counit $\varepsilon\colon FG \to 1_{\mathcal{D}}$ satisfying the two triangular identities, or via bijections $\mathcal{D}(F(C), D) \cong \mathcal{C}(C, G(D))$ natural in C and D.

3.1. THEOREM. *Let $G\colon \mathbb{D} \to \mathbb{C}$ be a* Pos-*functor between* Pos-*categories. For a lax*-Pos-*functor $F\colon \mathbb{C} \rightharpoonup \mathbb{D}$, the following statements are equivalent:*

1. *There exists a lax-natural* Pos-*transformation $\eta\colon 1_{\mathbb{C}} \rightharpoonup GF$ and a* Pos-*transformation $\varepsilon\colon FG \rightsquigarrow 1_{\mathbb{D}}$ such that $G(\varepsilon_D) \circ \eta_{G(D)} \leqslant 1_{G(D)}$ for all objects D in $\mathbb{D}$.*

2. *There exists a* Pos-*transformation $\rho = (\rho_{C,D})_{C,D}\colon \mathbb{D}(F(-),-) \rightsquigarrow \mathbb{C}(-,G(-))$ oplax-natural in C and natural in D together with a* Pos-*transformation $\sigma = (\sigma_{C,D})_{C,D}\colon \mathbb{C}(-,G(-)) \rightsquigarrow \mathbb{D}(F(-),-)$ lax-natural in C such that $\rho \circ \sigma \leqslant 1_{\mathbb{C}(-,G(-))}$.*

3. *There exists a* Pos-*transformation $\rho' = (\rho'_{C,D})_{C,D}\colon \mathbb{D}(F(-),-) \rightsquigarrow \mathbb{C}(-,G(-))$ oplax-natural in C and lax-natural in D together with a* Pos-*transformation $\sigma' = (\sigma'_{C,D})_{C,D}\colon \mathbb{C}(-,G(-)) \rightsquigarrow \mathbb{D}(F(-),-)$ such that $\rho' \circ \sigma' \leqslant 1_{\mathbb{C}(-,G(-))}$.*

PROOF. 1 $\Rightarrow$ 2: Given objects C in $\mathbb{C}$ and D in $\mathbb{D}$, we define

$$\rho_{C,D}(f) = G(f) \circ \eta_C \in \mathbb{C}(C, G(D))$$

for any morphism $f\colon F(C) \to D$ in $\mathbb{D}$. This defines an order-preserving function $\rho_{C,D}\colon \mathbb{D}(F(C),D) \to \mathbb{C}(C,G(D))$. The corresponding Pos-transformation $\rho\colon \mathbb{D}(F(-),-) \rightsquigarrow \mathbb{C}(-,G(-))$ is oplax-natural in C since G is a Pos-functor and η is lax-natural. It is natural in D again since G is a Pos-functor. Now, for a morphism $g\colon C \to G(D)$ in $\mathbb{C}$, we define

$$\sigma_{C,D}(g) = \varepsilon_D \circ F(g) \in \mathbb{D}(F(C), D).$$

The corresponding Pos-transformation $\sigma\colon \mathbb{C}(-,G(-)) \rightsquigarrow \mathbb{D}(F(-),-)$ is lax-natural in C since F is a lax-Pos-functor. Finally, given $g\colon C \to G(D)$, we have

$$\begin{aligned}
\rho_{C,D}(\sigma_{C,D}(g)) &= \rho_{C,D}(\varepsilon_D \circ F(g)) \\
&= G(\varepsilon_D \circ F(g)) \circ \eta_C \\
&= G(\varepsilon_D) \circ GF(g) \circ \eta_C \\
&\leqslant G(\varepsilon_D) \circ \eta_{G(D)} \circ g \\
&\leqslant g
\end{aligned}$$

proving 2.

2 $\Rightarrow$ 3 is obvious.

3 $\Rightarrow$ 1: Given such ρ' and σ', we define as usual

$$\eta_C = \rho'_{C,F(C)}(1_{F(C)})$$

for any object C of $\mathbb{C}$ and

$$\varepsilon_D = \sigma'_{G(D),D}(1_{G(D)})$$

for any object D of $\mathbb{D}$. The Pos-transformation $\eta\colon 1_{\mathbb{C}} \rightsquigarrow GF$ is lax-natural since, for $f\colon C \to C'$ in $\mathbb{C}$, we have

$$\begin{aligned} GF(f) \circ \eta_C &= GF(f) \circ \rho'_{C,F(C)}(1_{F(C)}) \\ &\leqslant \rho'_{C,F(C')}(F(f)) && \text{since } \rho' \text{ is lax-natural in } D \\ &\leqslant \rho'_{C',F(C')}(1_{F(C')}) \circ f && \text{since } \rho' \text{ is oplax-natural in } C \\ &= \eta_{C'} \circ f. \end{aligned}$$

Finally, for any object D in $\mathbb{D}$, we can compute

$$\begin{aligned} G(\varepsilon_D) \circ \eta_{G(D)} &= G(\sigma'_{G(D),D}(1_{G(D)})) \circ \rho'_{G(D),FG(D)}(1_{FG(D)}) \\ &\leqslant \rho'_{G(D),D}(\sigma'_{G(D),D}(1_{G(D)})) && \text{since } \rho' \text{ is lax-natural in } D \\ &\leqslant 1_{G(D)} \end{aligned}$$

proving 1. ■

3.2. Remark. In the above theorem, if $\mathbb{C}$ is of the form $\mathcal{P}_0(\mathcal{C})$ for a category $\mathcal{C}$, then the inequality $G(\varepsilon_D) \circ \eta_{G(D)} \leqslant 1_{G(D)}$ in 1 becomes an equality $G(\varepsilon_D) \circ \eta_{G(D)} = 1_{G(D)}$ for each object D in $\mathbb{D}$. Indeed, in that case, $1_{G(D)}$ is a singleton and the left hand side is a non-empty subset of it.

3.3. Definition. *Let $G\colon \mathbb{D} \to \mathbb{C}$ be a* Pos-*functor between* Pos-*categories. A* weak left adjoint *for G is a lax-*Pos-*functor $F\colon \mathbb{C} \rightharpoonup \mathbb{D}$ satisfying the equivalent conditions of Theorem 3.1.*

Note that in general, a Pos-functor may have many weak left adjoints. Our notion of weak adjoints generalizes the one in [17] where ordinary functors $G\colon \mathcal{D} \to \mathcal{C}$ were considered. There, weak left adjoints send a morphism of $\mathcal{C}$ to a non-empty set of morphisms in $\mathcal{D}$. Regarding the above definition for the Pos-functor $\mathcal{P}_0(G)\colon \mathcal{P}_0(\mathcal{D}) \to \mathcal{P}_0(\mathcal{C})$, a weak left adjoint for $\mathcal{P}_0(G)$ sends non-empty sets of morphisms of $\mathcal{C}$ to non-empty sets of morphisms in $\mathcal{D}$. In [16], a weak left adjoint for a functor $G\colon \mathcal{D} \to \mathcal{C}$ is defined as an (ordinary) functor $F\colon \mathcal{C} \to \mathcal{D}$ together with natural transformations $\eta\colon 1_{\mathcal{C}} \to GF$ and $\varepsilon\colon FG \to 1_{\mathcal{D}}$ satisfying only one triangular identity, namely $(1_G \star \varepsilon)(\eta \star 1_G) = 1_G$.

In the strong case, a left adjoint for a functor $G\colon \mathcal{D} \to \mathcal{C}$ exists if and only if each object in $\mathcal{C}$ has a reflection along G. Let us now study the weak version of this.

3.4. Definition. *Let $G\colon \mathbb{D} \to \mathbb{C}$ be a* Pos-*functor between* Pos-*categories. Given an object C in $\mathbb{C}$, a* weak reflection for C along G *is a pair (D,n) where D is an object of $\mathbb{D}$ and n a morphism $C \to G(D)$ in $\mathbb{C}$ such that for any morphism $f\colon C \to G(D')$ in $\mathbb{C}$, there exists a morphism $g\colon D \to D'$ in $\mathbb{D}$ such that $G(g) \circ n \leqslant f$.*

$$\begin{tikzcd} C \arrow[rr, "n"] \arrow[dr, "\forall f"'] & {} \arrow[d, phantom, "\geqslant"] & G(D) \arrow[dl, dotted, "\exists G(g)"] \\ & G(D') & \end{tikzcd}$$

When there is no ambiguity, we will sometimes denote the weak reflection simply by n. It is obvious to see that if $n' \colon C \to G(D)$ is a morphism such that $n' \leqslant n$, then n' is also a weak reflection for C along G if n is.

3.5. Example. Let $G \colon \mathcal{D} \to \mathcal{C}$ be an (ordinary) functor between categories and $C \in \mathcal{C}$ an object. A morphism $n \colon C \to G(D)$ in $\mathcal{C}$ is a weak reflection along G (in the classical sense) if and only if the singleton $\{n\} \colon C \to G(D)$ in $\mathcal{P}_0(\mathcal{C})$ is a weak reflection along $\mathcal{P}_0(G)$ in the sense of Definition 3.4. Indeed, if n is a weak reflection in $\mathcal{C}$ and $X \in \mathcal{P}_0(\mathbb{C}(C, G(D')))$, for each $f \in X$, we can choose $g_f \colon D \to D'$ in $\mathcal{D}$ such that $G(g_f)n = f$. The set $Y = \{g_f \,|\, f \in X\}$ then satisfies $\mathcal{P}_0(G)(Y) \circ \{n\} = X$. Conversely, if $\{n\}$ is a weak reflection along $\mathcal{P}_0(G)$ and $f \colon C \to G(D')$ is a morphism in $\mathcal{C}$, we know there exists $Y \in \mathcal{P}_0(\mathbb{D}(D, D'))$ such that $\mathcal{P}_0(G)(Y) \circ \{n\} \subseteq \{f\}$. Since Y is not empty, there exists $g \in Y$ which thus satisfies

$$\mathcal{P}_0(G)(\{g\}) \circ \{n\} \subseteq \mathcal{P}_0(G)(Y) \circ \{n\} \subseteq \{f\}.$$

Therefore $G(g)n = f$ and n is a weak reflection along G.

In addition, an object $C \in \mathcal{C}$ has a weak reflection along G if and only if it has a weak reflection along $\mathcal{P}_0(G)$. The ‘only if’ part immediately follows from the above. Conversely, if $N \in \mathcal{P}_0(\mathbb{C}(C, G(D)))$ is a weak reflection along $\mathcal{P}_0(G)$, then any singleton $\{n\} \subseteq N$ is also a weak reflection and thus any $n \in N$ is a weak reflection along G. Such a $n \in N$ exists since N is not empty.

3.6. Theorem. *Let $G \colon \mathbb{D} \to \mathbb{C}$ be an almost complete* Pos*-functor between almost complete* Pos*-categories. The following statements are equivalent:*

1. *Any object C in $\mathbb{C}$ has a weak reflection along G.*

2. *G has a weak left adjoint $F \colon \mathbb{C} \rightharpoonup \mathbb{D}$.*

Proof. 1 $\Rightarrow$ 2: For each object C in $\mathbb{C}$, we choose a weak reflection $\eta_C \colon C \to G(F(C))$. This already defines F on objects. Then, given $f \colon C \to C'$ in $\mathbb{C}$, we set

$$F(f) = \bigvee_{g \in Z_f} g$$

where

$$Z_f = \{g \colon F(C) \to F(C') \,|\, G(g) \circ \eta_C \leqslant \eta_{C'} \circ f\}$$

is a non-empty set since $(F(C), \eta_C)$ is a weak reflection. Clearly, F preserves the order of the hom-sets. Since $G(1_{F(C)}) \circ \eta_C = \eta_C$, we have $1_{F(C)} \in Z_{1_C}$ and thus $1_{F(C)} \leqslant F(1_C)$. Now, given $f \colon C \to C'$ and $f' \colon C' \to C''$ in $\mathbb{C}$, we have

$$F(f') \circ F(f) = \left(\bigvee_{g' \in Z_{f'}} g' \right) \circ \left(\bigvee_{g \in Z_f} g \right) = \bigvee_{(g,g') \in Z_f \times Z_{f'}} g'g \leqslant F(f'f)$$

where the second equality holds since $\mathbb{D}$ is almost complete and the inequality comes from the fact that, if $g \in Z_f$ and $g' \in Z_{f'}$,

$$G(g'g)\eta_C = G(g')G(g)\eta_C \leqslant G(g')\eta_{C'}f \leqslant \eta_{C''}f'f$$

implying that $g'g \in Z_{f'f}$. Hence, we have a lax-Pos-functor $F\colon \mathbb{C} \rightharpoonup \mathbb{D}$. The weak reflections η_C define a Pos-transformation $\eta\colon 1_{\mathbb{C}} \rightsquigarrow GF$. This η is lax-natural since, given $f\colon C \to C'$ in $\mathbb{C}$, we have

$$\begin{aligned}
GF(f) \circ \eta_C &= G(\bigvee_{g \in Z_f} g) \circ \eta_C \\
&= \left(\bigvee_{g \in Z_f} G(g) \right) \circ \eta_C && \text{since } G \text{ is almost complete} \\
&= \bigvee_{g \in Z_f} (G(g) \circ \eta_C) && \text{since } \mathbb{C} \text{ is almost complete} \\
&\leqslant \eta_{C'} \circ f.
\end{aligned}$$

Finally, for each object D in $\mathbb{D}$, using the fact that $\eta_{G(D)}$ is a weak reflection, we choose an $\varepsilon_D\colon FG(D) \to D$ such that $G(\varepsilon_D) \circ \eta_{G(D)} \leqslant 1_{G(D)}$. This defines the required Pos-transformation $\varepsilon\colon FG \rightsquigarrow 1_{\mathbb{D}}$ and F is a weak left adjoint for G.

2 $\Rightarrow$ 1: Let $F\colon \mathbb{C} \rightharpoonup \mathbb{D}$, $\eta\colon 1_{\mathbb{C}} \rightharpoonup GF$ and $\varepsilon\colon FG \rightsquigarrow 1_{\mathbb{D}}$ be given by Theorem 3.1. Let us prove that, for an object C in $\mathbb{C}$, $(F(C), \eta_C)$ is a weak reflection along G. Given any morphism $f\colon C \to G(D')$, the composite $\varepsilon_{D'} \circ F(f)\colon F(C) \to D'$ satisfies the required condition since

$$G(\varepsilon_{D'} \circ F(f)) \circ \eta_C = G(\varepsilon_{D'}) \circ GF(f) \circ \eta_C \leqslant G(\varepsilon_{D'}) \circ \eta_{G(D')} \circ f \leqslant f.$$

■

In [19], an (ordinary) functor $G\colon \mathcal{D} \to \mathcal{C}$ is said to 'have a weak left adjoint' if each object of $\mathcal{C}$ has a weak reflection along G, although no map $\mathcal{C} \to \mathcal{D}$ is considered. Due to Theorem 3.6, this terminology is thus consistent with ours.

4. Categories of (weak) fractions

Given a class Σ of morphisms in a category $\mathcal{C}$, the *category of fractions* of $\mathcal{C}$ with respect to Σ is, if it exists, the universal solution to the problem of finding a functor $F\colon \mathcal{C} \to \mathcal{D}$ which sends elements of Σ to isomorphisms [11]. When Σ is a set, the category of fractions always exists and its construction is recalled below. The main objective of this section is to develop a weak version of this. More precisely, if Σ is a set, we are going to construct a functor $F_{1/\Sigma}\colon \mathcal{C} \to \mathcal{C}[1/\Sigma]$ which sends elements of Σ to split monomorphisms and which is, in some sense, weakly universal with that property.

So let $\mathcal{C}$ be a category and Σ a set of morphisms in $\mathcal{C}$. We first construct a conditional graph $\mathcal{G}_{1/\Sigma}$ (in the sense of [4]; or more precisely a $\mathcal{V}$-conditional graph, for a bigger universe $\mathcal{V}$ containing our base universe $\mathcal{U}$). The objects of $\mathcal{G}_{1/\Sigma}$ are the same as the objects of $\mathcal{C}$ and given any pair (A, B) of objects, we define $\mathcal{G}_{1/\Sigma}(A, B)$ as the disjoint union

$$\mathcal{G}_{1/\Sigma}(A, B) = \mathcal{C}(A, B) \coprod \{\overline{s} \mid s \in \mathcal{C}(B, A) \cap \Sigma\}\,.$$

We consider the following commutativity conditions on $\mathcal{G}_{1/\Sigma}$ (i.e., the pairs of paths to quotient out in the path category of $\mathcal{G}_{1/\Sigma}$):

- $\overline{s} \circ s \approx 1_A$ for each $s \colon A \to B$ in Σ;
- $1_A \approx$ (empty path on A) for each object A in $\mathcal{C}$;
- $g \circ f \approx gf$ for each composable pair of arrows (f, g) in $\mathcal{C}$.

In the third family of commutativity conditions above, the left hand side $g \circ f$ represents the two arrow path made of f followed by g while the right hand side gf represents the single arrow path made of the composition of f and g in $\mathcal{C}$. The path category of $\mathcal{G}_{1/\Sigma}$ will be denoted by $\mathcal{C}[1/\Sigma]$. It is not hard to see that, since Σ is a set, $\mathcal{C}[1/\Sigma]$ is actually an ordinary $\mathcal{U}$-category. It comes equipped with a functor

$$\begin{aligned} F_{1/\Sigma} \colon \mathcal{C} &\longrightarrow \mathcal{C}[1/\Sigma] \\ A &\longmapsto A \\ f &\longmapsto [f] \end{aligned}$$

where $[p]$ denotes the equivalence class of a path p from $\mathcal{G}_{1/\Sigma}$. This functor $F_{1/\Sigma}$ sends elements of Σ to split monomorphisms since $[\overline{s}] \circ F_{1/\Sigma}(s) = 1_A$ for each $s \colon A \to B$ in Σ. Moreover, as attested by the Theorem 4.1 below, $F_{1/\Sigma}$ is, in the Pos-category world, weakly universal with that property.

Before proving this theorem, let us recall that to construct the category of fractions for $\mathcal{C}$ with respect to Σ, one first defines the conditional $\mathcal{V}$-graph $\mathcal{G}_{\Sigma^{-1}}$, which is just $\mathcal{G}_{1/\Sigma}$ together with the additional commutativity conditions

- $s \circ \overline{s} \approx 1_B$ for each $s \colon A \to B$ in Σ.

The path category of $\mathcal{G}_{\Sigma^{-1}}$ is denoted $\mathcal{C}[\Sigma^{-1}]$ and the obvious functor $F_{\Sigma^{-1}} \colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ is the expected category of fractions.

4.1. Theorem. *Let $\mathcal{C}$ be a category and Σ a set of morphisms in $\mathcal{C}$. Let also $\mathbb{D}$ be an almost complete* Pos-*category.*

1. *If $G \colon \mathcal{P}_0(\mathcal{C}) \rightharpoonup \mathbb{D}$ is an almost complete lax-*Pos-*functor such that*
   (A) *for all $s \colon A \to B$ in Σ, there exists $d \in \mathbb{D}(G(B), G(A))$ such that $d \circ G(sf) \leqslant G(f)$ for any $f \in \mathcal{C}(C, A)$,*

(B) if $G(sf) = G(sf')$ for some $f, f' \colon C \to A$ in $\mathcal{C}$ and $s \colon A \to B$ in Σ, then $G(hf) = G(hf')$ for any $h \in \mathcal{C}(A, D)$;

*then, there exists an almost complete lax-*Pos*-functor $H \colon \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \rightharpoonup \mathbb{D}$ such that $H \circ \mathcal{P}_0(F_{1/\Sigma}) = G$.*

2. *If $G \colon \mathcal{P}_0(\mathcal{C}) \to \mathbb{D}$ is an almost complete* Pos*-functor such that*

(C) for all $s \colon A \to B$ in Σ, there exists $d \in \mathbb{D}(G(B), G(A))$ such that $d \circ G(s) = 1_{G(A)}$;

then, G automatically satisfies conditions (A) and (B) and there exists an almost complete Pos*-functor $H \colon \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \to \mathbb{D}$ such that $H \circ \mathcal{P}_0(F_{1/\Sigma}) = G$.*

$$\begin{tikzcd} \mathcal{P}_0(\mathcal{C}) \arrow[rr, "\mathcal{P}_0(F_{1/\Sigma})"] \arrow[dr, "G"'] & & \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \arrow[dl, dotted, harpoon, "H"] \\ & \mathbb{D} & \end{tikzcd}$$

Proof. 1: On objects, H has to be defined as $H(A) = G(A)$. For $s \colon A \to B$ in Σ, we denote

$$D_s = \{d \in \mathbb{D}(G(B), G(A)) \,|\, d \circ G(sf) \leqslant G(f) \text{ for any } f \in \mathcal{C}(C, A)\}$$

which is not empty by our hypothesis (A) on G. Then, for a path $p \colon A \to B$ in $\mathcal{G}_{1/\Sigma}$, we set

$$\begin{aligned} H([p]) &= \\ &\bigvee_{[f_n \circ \overline{s_{n-1}} \circ \cdots \circ f_2 \circ \overline{s_1} \circ f_1] = [p]} \left(G(f_n) \circ \left(\bigvee_{d_{n-1} \in D_{s_{n-1}}} d_{n-1} \right) \circ \cdots \circ G(f_2) \circ \left(\bigvee_{d_1 \in D_{s_1}} d_1 \right) \circ G(f_1) \right) \\ &= \bigvee_{[f_n \circ \overline{s_{n-1}} \circ \cdots \circ f_2 \circ \overline{s_1} \circ f_1] = [p]} \Big(G(f_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ G(f_2) \circ \bigvee D_{s_1} \circ G(f_1) \Big) \end{aligned}$$

where the suprema is taken over all alternating paths $f_n \circ \overline{s_{n-1}} \circ \cdots \circ f_2 \circ \overline{s_1} \circ f_1$ equivalent to p with $n \geqslant 1$. It follows then that for each morphism f in $\mathcal{C}$, one has $G(f) \leqslant H([f])$, proving at the same time that $H(1_A) \geqslant 1_{H(A)}$ for each object A of $\mathcal{C}$. We then define

$$H(X) = \bigvee_{[p] \in X} H([p])$$

for any $X \in \mathcal{P}_0(\mathcal{C}[1/\Sigma](A, B))$. This turns H into an almost complete lax-Pos-functor. Indeed, since $\mathbb{D}$ is almost complete, it suffices to prove $H([q]) \circ H([p]) \leqslant H([q \circ p])$ for composable paths p and q. Again since $\mathbb{D}$ is almost complete, this follows from

$$\begin{aligned}
& G(g_m) \circ \bigvee D_{t_{m-1}} \circ \cdots \circ G(g_2) \circ \bigvee D_{t_1} \circ G(g_1) \circ G(f_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \\
& \qquad \circ G(f_2) \circ \bigvee D_{s_1} \circ G(f_1) \\
& \leqslant G(g_m) \circ \bigvee D_{t_{m-1}} \circ \cdots \circ G(g_2) \circ \bigvee D_{t_1} \circ G(g_1 f_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \\
& \qquad \circ G(f_2) \circ \bigvee D_{s_1} \circ G(f_1) \\
& \leqslant H([p \circ q])
\end{aligned}$$

where $f_n \circ \overline{s_{n-1}} \circ \cdots \circ f_2 \circ \overline{s_1} \circ f_1$ and $g_m \circ \overline{t_{m-1}} \circ \cdots \circ g_2 \circ \overline{t_1} \circ g_1$ are representative of $[p]$ and $[q]$ respectively. For the first part of the statement, since G is almost complete, it remains now to prove that $H([f]) \leqslant G(f)$ for any morphism $f \colon A \to B$ in $\mathcal{C}$.

In order to do so, we are going to prove that for each path of the form $g_n \circ \overline{s_{n-1}} \circ \cdots \circ g_2 \circ \overline{s_1} \circ g_1$ equivalent to f, there exist morphisms $h_1, \ldots, h_n$ satisfying

- $h_1 = 1_A$,
- $G(g_i h_i) = G(s_i h_{i+1})$ for all $1 \leqslant i < n$,
- $G(g_n h_n) = G(f)$.

$$\begin{tikzcd}
 & B_1 \arrow[dr, dotted, "\overline{s_1}"] & & B_2 \arrow[dr, dotted, "\overline{s_2}"] & & \cdots & B_{n-1} \arrow[dr, dotted, "\overline{s_{n-1}}"] & & B=B_n \\
A=A_1 \arrow[loop left, "h_1"] \arrow[ur, "g_1"] \arrow[rr, "h_2"'] \arrow[rrrr, bend right, "h_3"'] \arrow[rrrrrrr, bend right, "h_n"'] & & A_2 \arrow[ul, "s_1"] \arrow[ur, "g_2"] & & A_3 \arrow[ul, "s_2"] & \cdots & & A_n \arrow[ul, "s_{n-1}"] \arrow[ur, "g_n"] &
\end{tikzcd}$$

Firstly, we note that if the path is f itself (i.e., $n = 1$ and $g_1 = f$), the claim is trivial. Now, suppose those morphisms $h_1, \ldots, h_n$ exist for the path $g_n \circ \overline{s_{n-1}} \circ \cdots \circ g_1$ and let us construct the desired morphisms $k_1, \ldots, k_{n+1}$ for the path

$$g_n \circ \overline{s_{n-1}} \circ \cdots \overline{s_m} \circ y \circ \overline{t} \circ (tx) \circ \overline{s_{m-1}} \circ \cdots \circ \overline{s_1} \circ g_1$$

where $1 \leqslant m \leqslant n$, $t \in \Sigma$ and $yx = g_m$ (i.e., g_m has been replaced by $y \circ \overline{t} \circ (tx)$).

$$\begin{tikzcd}
 & A_m \arrow[rr, "g_m"] \arrow[dl, "s_{m-1}"'] \arrow[dr, "x"] & & B_m \arrow[dr, dotted, "\overline{s_m}"] & \\
B_{m-1} \arrow[ur, dotted, "\overline{s_{m-1}}"] & & X \arrow[ur, "y"] \arrow[d, shift right, "t"'] & & A_{m+1} \arrow[ul, "s_m"] \\
 & & Z \arrow[u, shift right, dotted, "\overline{t}"'] & &
\end{tikzcd}$$

We set

- $k_i = h_i$ for all $1 \leqslant i \leqslant m$,
- $k_{m+1} = xh_m$,
- $k_i = h_{i-1}$ for all $m+1 < i \leqslant n+1$,

and all the desired equalities for the k_i's immediately follow from the ones for the h_i's.

Suppose now the morphisms $h_1, \dots, h_n$ exist for the path $g_n \circ \overline{s_{n-1}} \circ \cdots \circ g_1$ and let us construct the desired morphisms $k_1, \dots, k_{n-1}$ for the path

$$g_n \circ \overline{s_{n-1}} \circ \cdots \overline{s_{m+1}} \circ (g_{m+1}x) \circ \overline{s_{m-1}} \circ \cdots \circ \overline{s_1} \circ g_1$$

where $1 \leqslant m < n$ and $s_m x = g_m$ (i.e., $g_{m+1} \circ \overline{s_m} \circ g_m$ has been replaced by $g_{m+1}x$).

$$\begin{tikzcd}
 & B_m \arrow[dr, dotted, bend left=10, "\overline{s_m}"] & & B_{m+1} \\
A_m \arrow[ur, "g_m"] \arrow[rr, "x"'] & & A_{m+1} \arrow[ul, "s_m"] \arrow[ur, "g_{m+1}"'] &
\end{tikzcd}$$

We set

- $k_i = h_i$ for all $1 \leqslant i \leqslant m$,
- $k_i = h_{i+1}$ for all $m < i < n$.

All the desired equalities for the k_i's immediately follow from the ones for the h_i's except for $G(g_{m+1}xh_m) = G(s_{m+1}h_{m+2})$ if $m < n-1$ and $G(g_n x h_{n-1}) = G(f)$ if $m = n-1$. Since $G(s_{m+1}h_{m+2}) = G(g_{m+1}h_{m+1})$ for the first case and $G(f) = G(g_n h_n) = G(g_{m+1}h_{m+1})$ in the second, it remains to prove $G(g_{m+1}xh_m) = G(g_{m+1}h_{m+1})$. Using our hypothesis (B) about G, it suffices to notice that

$$G(s_m x h_m) = G(g_m h_m) = G(s_m h_{m+1}).$$

Since each path equivalent to f is obtained from f by applying a finite number of times these two operations, this proves the claim that such morphisms $h_1, \dots, h_n$ always exist. To prove that $H([f]) \leqslant G(f)$, by definition of $H([f])$, we need to prove that $G(g_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ \bigvee D_{s_1} \circ G(g_1) \leqslant G(f)$ for all path $g_n \circ \overline{s_{n-1}} \circ \cdots \circ \overline{s_1} \circ g_1$ equivalent to f. Using our morphisms $h_1, \dots, h_n$, we have for each $1 \leqslant i < n$:

$$\begin{aligned}
&G(g_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ G(g_{i+1}) \circ \bigvee D_{s_i} \circ G(g_i h_i) \\
&= G(g_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ G(g_{i+1}) \circ \bigvee D_{s_i} \circ G(s_i h_{i+1}) \\
&\leqslant G(g_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ G(g_{i+1}) \circ G(h_{i+1}) && \text{since } \mathbb{D} \text{ is almost complete} \\
&\leqslant G(g_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ G(g_{i+1}h_{i+1}).
\end{aligned}$$

Therefore, we have

$$\begin{aligned} G(g_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ \bigvee D_{s_1} \circ G(g_1) &= G(g_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ \bigvee D_{s_1} \circ G(g_1 h_1) \\ &\leqslant G(g_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ \bigvee D_{s_2} \circ G(g_2 h_2) \\ &\leqslant \cdots \\ &\leqslant G(g_n h_n) \\ &= G(f) \end{aligned}$$

concluding the proof of the first part of the statement.

2: Suppose now that G is an almost complete Pos-functor satisfying condition (C). It is easy to see that G satisfies conditions (A) and (B). Notice also that, for any $s\colon A \to B$ in Σ,

$$D_s = \{d \in \mathbb{D}(G(B), G(A)) \mid d \circ G(s) \leqslant 1_{G(A)}\}$$

and $\bigvee D_s \circ G(s) = 1_{G(A)}$. It remains now to prove that the above constructed H is also a Pos-functor. Since H is almost complete, it suffices to prove $H([q \circ p]) = H([q]) \circ H([p])$ for any pair of composable paths (p, q). In order to do so, let us define, for each path p, the morphism $H'(p)$ in $\mathbb{D}$ via the following rules:

- $H'(\text{empty path on } A) = 1_{G(A)}$ for any object A,
- $H'(f \circ p) = G(f) \circ H'(p)$ for any path p composable with the morphism f of $\mathcal{C}$,
- $H'(\overline{s} \circ p) = \bigvee D_s \circ H'(p)$ for any $s\colon A \to B$ in Σ and any path p of codomain B.

We can immediately notice that since G is a Pos-functor:

- $H'(\overline{s} \circ s) = \bigvee D_s \circ G(s) = 1_{G(A)} = H'(1_A)$ for any $s\colon A \to B$ in Σ,
- $H'(1_A) = H'(\text{empty path on } A)$ for any object A,
- $H'(g \circ f) = G(g) \circ G(f) = G(gf) = H'(gf)$ for any composable pair of arrows (f, g),
- $H'(q \circ p) = H'(q) \circ H'(p)$ for any composable pair of paths (p, q) (by induction on the size of q).

This proves that $H'(p) = H'(p')$ for any pair of equivalent paths (p, p'). Therefore, for any path p, we have

$$\begin{aligned} H([p]) &= \bigvee_{[f_n \circ \overline{s_{n-1}} \circ \cdots \circ f_2 \circ \overline{s_1} \circ f_1] = [p]} \Big(G(f_n) \circ \bigvee D_{s_{n-1}} \circ \cdots \circ G(f_2) \circ \bigvee D_{s_1} \circ G(f_1) \Big) \\ &= \bigvee_{[f_n \circ \overline{s_{n-1}} \circ \cdots \circ f_2 \circ \overline{s_1} \circ f_1] = [p]} H'(f_n \circ \overline{s_{n-1}} \circ \cdots \circ f_2 \circ \overline{s_1} \circ f_1) \\ &= \bigvee_{[f_n \circ \overline{s_{n-1}} \circ \cdots \circ f_2 \circ \overline{s_1} \circ f_1] = [p]} H'(p) \\ &= H'(p). \end{aligned}$$

So, for any pair of composable paths (p, q), we have that

$$H([q \circ p]) = H'(q \circ p) = H'(q) \circ H'(p) = H([q]) \circ H([p])$$

as desired. ■

4.2. Remark. Although condition (A) is necessary for the existence of a factorization $H\colon \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \rightharpoonup \mathbb{D}$, condition (B) seems not to be so in the 'lax world'. However, it mimics the condition that $G(s)$ should be a monomorphism for $s \in \Sigma$.

4.3. Remark. In [15], given a category $\mathcal{C}$ with pullbacks and a Σ containing isomorphisms and stable under composition and pullbacks, the authors construct a functor $\Phi\colon \mathcal{C} \to \mathrm{Sect}(\mathcal{C}, \Sigma)$ which sends elements of Σ to split monomorphisms. Moreover, a *functorial* choice of retractions is given, and Φ is (strongly) universal among those functors equipped with such a functorial choice of retractions (and satisfying a Beck-Chevalley condition). In our category $\mathcal{C}[1/\Sigma]$, such a functorial choice is not possible since in general $\overline{s} \circ \overline{t} \neq \overline{ts}$ for $s, t, ts \in \Sigma$.

5. The Orthogonal Subcategory Problem

We use the following standard definitions (see for instance [14]).

5.1. Definition. *In a category $\mathcal{C}$, an object I is* injective with respect to *(respectively,* orthogonal to*) a morphism $s\colon A \to B$ if for all morphism $f\colon A \to I$, there exists (respectively, there exists a unique) morphism $g\colon B \to I$ such that $gs = f$.*

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "\forall f"'] & B \arrow[dl, dotted, "\exists g\ (resp.\ \exists! g)"] \\ I & \end{tikzcd}$$

In this case, we write $s \vartriangle I$ (respectively $s \perp I$).

For a class Σ of morphisms in $\mathcal{C}$, we write

$$\Sigma^{\vartriangle} = \{I \in \mathrm{ob}(\mathcal{C}) \,|\, s \vartriangle I \text{ for all } s \in \Sigma\},$$

$$\Sigma^{\perp} = \{I \in \mathrm{ob}(\mathcal{C}) \,|\, s \perp I \text{ for all } s \in \Sigma\}$$

and for a class $\mathcal{I}$ of objects in $\mathcal{C}$, we write

$$\mathcal{I}^{\triangledown} = \{s \in \mathcal{C} \,|\, s \vartriangle I \text{ for all } I \in \mathcal{I}\},$$

$$\mathcal{I}^{\top} = \{s \in \mathcal{C} \,|\, s \perp I \text{ for all } I \in \mathcal{I}\}.$$

Given a class of morphisms Σ in $\mathcal{C}$, the Orthogonal Subcategory Problem (OSP), which first appeared in [10], deals with the question to know whether $\Sigma^{\perp}$, considered as a full subcategory of $\mathcal{C}$, is reflective in $\mathcal{C}$. This problem has been deeply studied (see for instance [1, 3, 4, 6, 18, 20]) and connections with the category of fractions $\mathcal{C}[\Sigma^{-1}]$ has been established [11]. We recall in this section these connections. Since in [11] the condition that Σ admits a left calculus of fractions was unnecessarily used, and since it is not known to the author whether this easy generalization already appears in the literature, we re-prove these results here without the left calculus of fractions assumption. The final goal is to consider the weak version of Theorem 5.3 in the next section. We start with a lemma generalizing 4.1.2 in [11]. To avoid technical size issues, we assume that Σ is a set although this assumption might be relaxed.

5.2. Lemma. *Let Σ be a set of morphisms in a category $\mathcal{C}$. Given any map $[p]\colon A \to K$ in $\mathcal{C}[\Sigma^{-1}]$ where $K \in \Sigma^{\perp}$, there exists a unique morphism $f\colon A \to K$ in $\mathcal{C}$ such that $F_{\Sigma^{-1}}(f) = [p]$.*

Proof. Let us first prove existence. The map $[p]\colon A \to K$ is represented by a path $f_n \circ \overline{s_{n-1}} \circ \cdots \circ \overline{s_1} \circ f_1$ in $\mathcal{G}_{\Sigma^{-1}}$. We are going to construct morphisms $g_i\colon B_i \to K$ in $\mathcal{C}$ for each $1 \leqslant i \leqslant n$ satisfying

- $g_n = 1_K$ and
- $g_i s_i = g_{i+1} f_{i+1}$ for all $1 \leqslant i < n$.

$$\begin{tikzcd}
A{=}A_1 \arrow[dr, "f_1"'] & & A_2 \arrow[dl, "s_1"] \arrow[dr, "f_2"'] & & \cdots & & A_n \arrow[dl, "s_{n-1}"] \arrow[dr, "f_n"] & \\
& B_1 \arrow[ur, dotted, "\overline{s_1}"] \arrow[rrrrrr, bend right, "g_1"'] & & B_2 \arrow[rrrr, bend right, "g_2"'] & \cdots & B_{n-1} \arrow[ur, dotted, "\overline{s_{n-1}}"] \arrow[rr, "g_{n-1}"'] & & B_n{=}K \arrow[loop right, "g_n"]
\end{tikzcd}$$

We first construct $g_n = 1_K$. Now, suppose g_{i+1} is constructed for some $1 \leqslant i < n$. Since $K \in \Sigma^{\perp}$ and $s_i \in \Sigma$, there exists a unique morphism $g_i\colon B_i \to K$ such that $g_i s_i = g_{i+1} f_{i+1}$. We now claim that $F_{\Sigma^{-1}}(g_1 f_1) = [p]$, proving the existence:

$$\begin{aligned}
&[p] = [f_n] \circ \overline{s_{n-1}} \circ \cdots \circ \overline{s_1} \circ [f_1] \\
&= F_{\Sigma^{-1}}(g_n f_n) \circ F_{\Sigma^{-1}}(s_{n-1})^{-1} \circ F_{\Sigma^{-1}}(f_{n-1}) \circ \cdots \circ F_{\Sigma^{-1}}(f_2) \circ F_{\Sigma^{-1}}(s_1)^{-1} \circ F_{\Sigma^{-1}}(f_1) \\
&= F_{\Sigma^{-1}}(g_{n-1} s_{n-1}) \circ F_{\Sigma^{-1}}(s_{n-1})^{-1} \circ F_{\Sigma^{-1}}(f_{n-1}) \circ \cdots \circ F_{\Sigma^{-1}}(f_2) \circ F_{\Sigma^{-1}}(s_1)^{-1} \circ F_{\Sigma^{-1}}(f_1) \\
&= F_{\Sigma^{-1}}(g_{n-1} f_{n-1}) \circ F_{\Sigma^{-1}}(s_{n-2})^{-1} \circ F_{\Sigma^{-1}}(f_{n-2}) \circ \cdots F_{\Sigma^{-1}}(f_2) \circ F_{\Sigma^{-1}}(s_1)^{-1} \circ F_{\Sigma^{-1}}(f_1) \\
&= \cdots \\
&= F_{\Sigma^{-1}}(g_1 f_1)
\end{aligned}$$

For uniqueness, we consider the full inclusion $I\colon \Sigma^{\perp} \hookrightarrow \mathcal{C}$ and its shape $\mathcal{V}$-category [8] $\mathcal{S}_I$ for a bigger universe $\mathcal{V} \ni \mathcal{U}$. This $\mathcal{S}_I$ has the same objects as $\mathcal{C}$, and morphisms $A \to B$ in $\mathcal{S}_I$ are natural transformations $\mathcal{C}(B, I(-)) \Rightarrow \mathcal{C}(A, I(-))$. Composition is simply the composition of natural transformations. The functor

$$\begin{aligned} D\colon \mathcal{C} &\longrightarrow \mathcal{S}_I \\ A &\longmapsto A \\ f &\longmapsto \mathcal{C}(f, I(-)) \end{aligned}$$

sends elements of Σ to isomorphisms by definition of $\Sigma^\perp$. Therefore, D factorizes uniquely through $F_{\Sigma^{-1}}$. So, if $f, f'\colon A \to K$ are such that $F_{\Sigma^{-1}}(f) = F_{\Sigma^{-1}}(f')$, then $D(f) = D(f')$. As usual, this implies

$$f = D(f)_K(1_K) = D(f')_K(1_K) = f'$$

proving uniqueness. ■

5.3. Theorem. *Let Σ be a set of morphisms in a category $\mathcal{C}$. Denote by $I\colon \Sigma^\perp \hookrightarrow \mathcal{C}$ the full inclusion functor and $F_{\Sigma^{-1}}\colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ the category of fractions with respect to Σ.*

$$\begin{tikzcd} & \Sigma^\perp \arrow[dl, hook', "I"'] \arrow[dr, "F_{\Sigma^{-1}} \circ I"] & \\ \mathcal{C} \arrow[rr, "F_{\Sigma^{-1}}"'] & & \mathcal{C}[\Sigma^{-1}] \end{tikzcd}$$

The following statements are equivalent:

(i) I has a left adjoint R for which the unit is inverted by $F_{\Sigma^{-1}}$;

(ii) $F_{\Sigma^{-1}}$ has a fully faithful right adjoint;

(iii) $F_{\Sigma^{-1}} \circ I$ is essentially surjective on objects;

(iv) $F_{\Sigma^{-1}} \circ I$ is an equivalence.

In this case, we have

$$\{f \in \mathcal{C} \mid F_{\Sigma^{-1}}(f) \text{ is an isomorphism}\} = \{f \in \mathcal{C} \mid R(f) \text{ is an isomorphism}\} = \Sigma^{\perp\top}$$

and this class of morphisms admits a left calculus of fractions as introduced in [11].

Proof. We first suppose (i). Let $R\colon \mathcal{C} \to \Sigma^\perp$ be the left adjoint of I, $\eta\colon 1_{\mathcal{C}} \Rightarrow IR$ the unit and $\varepsilon\colon RI \Rightarrow 1_{\Sigma^\perp}$ the counit. Since I is fully faithful, ε is an isomorphism. Moreover, by assumption, $1_{F_{\Sigma^{-1}}} \star \eta$ is an isomorphism. Given any $s\colon A \to B$ in $\Sigma^{\perp\top}$,

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \arrow[dl, dotted, "f"'] \\ R(A) \arrow[r, shift left, "R(s)"] & R(B) \arrow[l, shift left, dotted, "g"] \end{tikzcd}$$

since $R(A) \in \Sigma^\perp$, there exists a unique $f\colon B \to R(A)$ such that $fs = \eta_A$. Moreover, since $R(B) \in \Sigma^\perp$, $R(s)f = \eta_B$. Using the universality of η_B, there exists a unique $g\colon R(B) \to R(A)$ such that $g\eta_B = f$. These identities imply $gR(s)\eta_A = \eta_A$ and $R(s)g\eta_B = \eta_B$, which in turn imply that $R(s)$ is an isomorphism with inverse g. This proves $\Sigma^{\perp\top} \subseteq \{f \in \mathcal{C} \,|\, R(f) \text{ is an isomorphism}\}$. Conversely, if $s\colon A \to B$ is such that $R(s)$ is an isomorphism and $h\colon A \to K$ is a map with codomain $K \in \Sigma^\perp$,

$$\begin{tikzcd} & A \arrow[r, "s"] \arrow[dl, "h"'] \arrow[d, "\eta_A"] & B \arrow[d, "\eta_B"] \\ K & R(A) \arrow[l, dotted, "k"] \arrow[r, "\cong", "R(s)"'] & R(B) \end{tikzcd}$$

there exists a unique $k\colon R(A) \to K$ such that $k\eta_A = h$. So $kR(s)^{-1}\eta_B s = h$. For uniqueness, if $x\colon B \to K$ is such that $xs = h$, there exists a unique $y\colon R(B) \to K$ with $y\eta_B = x$. But $yR(s)\eta_A = h = k\eta_A$ implies that $yR(s) = k$ and so $x = y\eta_B = kR(s)^{-1}\eta_B$. We have thus proved that

$$\{f \in \mathcal{C} \,|\, R(f) \text{ is an isomorphism}\} = \Sigma^{\perp\top}$$

and in particular elements of Σ are sent by R to isomorphisms. Therefore, there is a unique functor $\hat{R}\colon \mathcal{C}[\Sigma^{-1}] \to \Sigma^\perp$ such that $\hat{R} \circ F_{\Sigma^{-1}} = R$.

$$\begin{tikzcd} & \Sigma^\perp \arrow[dl, hook, "I"', shift right] \arrow[dr, "F_{\Sigma^{-1}} \circ I", shift left] & \\ \mathcal{C} \arrow[ur, "R"', shift right] \arrow[ur, phantom, "\top"] \arrow[rr, "F_{\Sigma^{-1}}"'] & & \mathcal{C}[\Sigma^{-1}] \arrow[ul, "\hat{R}", shift left] \end{tikzcd}$$

Using again the universal property of $F_{\Sigma^{-1}}$, there exists a unique natural transformation $\alpha\colon F_{\Sigma^{-1}} \circ I \circ \hat{R} \Rightarrow 1_{\mathcal{C}[\Sigma^{-1}]}$ such that $\alpha \star 1_{F_{\Sigma^{-1}}} = (1_{F_{\Sigma^{-1}}} \star \eta)^{-1}$. Moreover, since $F_{\Sigma^{-1}}$ is surjective on objects, α itself is an isomorphism. But we also have $\hat{R} \circ F_{\Sigma^{-1}} \circ I = R \circ I \cong 1_{\Sigma^\perp}$, proving that $F_{\Sigma^{-1}} \circ I$ is an equivalence with pseudo-inverse $\hat{R}$. Then, $F_{\Sigma^{-1}} \cong (F_{\Sigma^{-1}} \circ I) \circ R$ has $I \circ \hat{R}$ as right adjoint which is fully faithful since I and $\hat{R}$ are. We have thus proved (i) $\Rightarrow$ (ii) + (iii) + (iv). Moreover, $\hat{R}$ being an equivalence, we already know that

$$\{f \in \mathcal{C} \,|\, F_{\Sigma^{-1}}(f) \text{ is an isomorphism}\} = \{f \in \mathcal{C} \,|\, R(f) \text{ is an isomorphism}\} = \Sigma^{\perp\top}.$$

By Proposition 5.3.1 in [4], this class of morphisms admits a left calculus of fractions.

Assume now (ii). Let $G\colon \mathcal{C}[\Sigma^{-1}] \to \mathcal{C}$ be the fully faithful right adjoint of $F_{\Sigma^{-1}}$ with unit $\eta\colon 1_{\mathcal{C}} \Rightarrow G \circ F_{\Sigma^{-1}}$ and isomorphism counit $\alpha\colon F_{\Sigma^{-1}} \circ G \Rightarrow 1_{\mathcal{C}[\Sigma^{-1}]}$. We now prove that for each object C of $\mathcal{C}[\Sigma^{-1}]$ (i.e., of $\mathcal{C}$), $G(C) \in \Sigma^\perp$. This would prove (ii) $\Rightarrow$ (iii). So we consider $s\colon A \to B$ in Σ and $f\colon A \to G(C)$ any morphism in $\mathcal{C}$. By the universal property of α_C, we know there exists a unique $g\colon B \to G(C)$ such that $\alpha_C F_{\Sigma^{-1}}(g) = \alpha_C F_{\Sigma^{-1}}(f) F_{\Sigma^{-1}}(s)^{-1}$.

$$\begin{tikzcd}
F_{\Sigma^{-1}}(B) \arrow[r, "F_{\Sigma^{-1}}(s)^{-1}"] \arrow[d, dotted, "F_{\Sigma^{-1}}(g)"'] & F_{\Sigma^{-1}}(A) \arrow[r, "F_{\Sigma^{-1}}(f)"] & F_{\Sigma^{-1}}G(C) \arrow[d, "\alpha_C"] \\
F_{\Sigma^{-1}}G(C) \arrow[rr, "\alpha_C"'] & & C
\end{tikzcd}$$

But this identity is equivalent to $\alpha_C F_{\Sigma^{-1}}(gs) = \alpha_C F_{\Sigma^{-1}}(f)$, which, again by the universal property of α_C, is equivalent to $gs = f$, proving that $G(C) \in \Sigma^{\perp}$ and so (ii) $\Rightarrow$ (iii).

By Lemma 5.2, we know that $F_{\Sigma^{-1}} \circ I$ is fully faithful, proving the equivalence (iii) $\Leftrightarrow$ (iv) (under the axiom of choice).

We now prove (iv) $\Rightarrow$ (i). So suppose we have a pseudo-inverse $H\colon \mathcal{C}[\Sigma^{-1}] \to \Sigma^{\perp}$ for $F_{\Sigma^{-1}} \circ I$ equipped with natural isomorphisms $\alpha\colon F_{\Sigma^{-1}} \circ I \circ H \Rightarrow 1_{\mathcal{C}[\Sigma^{-1}]}$ and $\varepsilon\colon H \circ F_{\Sigma^{-1}} \circ I \Rightarrow 1_{\Sigma^{\perp}}$ satisfying $1_H \star \alpha = \varepsilon \star 1_H$ and $\alpha \star 1_{F_{\Sigma^{-1}} \circ I} = 1_{F_{\Sigma^{-1}} \circ I} \star \varepsilon$. For each object C in $\mathcal{C}$,

$$\alpha^{-1}_{F_{\Sigma^{-1}}(C)}\colon F_{\Sigma^{-1}}(C) \to F_{\Sigma^{-1}} I H F_{\Sigma^{-1}}(C)$$

is a morphism in $\mathcal{C}[\Sigma^{-1}]$ whose codomain is in $\Sigma^{\perp}$. By Lemma 5.2, there exists a unique morphism $\eta_C\colon C \to IHF_{\Sigma^{-1}}(C)$ in $\mathcal{C}$ such that $F_{\Sigma^{-1}}(\eta_C) = \alpha^{-1}_{F_{\Sigma^{-1}}(C)}$. The naturality of α and the uniqueness part of Lemma 5.2 show that this defines a natural transformation $\eta\colon 1_{\mathcal{C}} \Rightarrow I \circ H \circ F_{\Sigma^{-1}}$ inverted by $F_{\Sigma^{-1}}$. We then have

$$(\varepsilon \star 1_{H \circ F_{\Sigma^{-1}}})(1_{H \circ F_{\Sigma^{-1}}} \star \eta) = (\varepsilon \star 1_{H \circ F_{\Sigma^{-1}}})(1_H \star \alpha^{-1} \star 1_{F_{\Sigma^{-1}}}) = 1_{H \circ F_{\Sigma^{-1}}}$$

and

$$(1_{F_{\Sigma^{-1}} \circ I} \star \varepsilon)(1_{F_{\Sigma^{-1}}} \star \eta \star 1_I) = (1_{F_{\Sigma^{-1}} \circ I} \star \varepsilon)(\alpha^{-1} \star 1_{F_{\Sigma^{-1}} \circ I}) = 1_{F_{\Sigma^{-1}} \circ I}$$

proving once again by Lemma 5.2 that $(1_I \star \varepsilon)(\eta \star 1_I) = 1_I$. Therefore I has a left adjoint $H \circ F_{\Sigma^{-1}}$ with unit η inverted by $F_{\Sigma^{-1}}$. ■

In general, $\Sigma^{\perp}$ may be reflexive in $\mathcal{C}$ without the unit being inverted by $F_{\Sigma^{-1}}$ (an example can be found in [7]). But in that case, we have the following corollary.

5.4. Corollary. *Let Σ be a set of morphisms in a category $\mathcal{C}$ such that $\Sigma^{\perp\top}$ is also a set. Denote by $I\colon \Sigma^{\perp} \hookrightarrow \mathcal{C}$ the full inclusion functor and $F_{(\Sigma^{\perp\top})^{-1}}\colon \mathcal{C} \to \mathcal{C}[(\Sigma^{\perp\top})^{-1}]$ the category of fractions with respect to $\Sigma^{\perp\top}$.*

$$\begin{tikzcd}
 & \Sigma^{\perp} \arrow[dl, hook', "I"'] \arrow[dr, "F_{(\Sigma^{\perp\top})^{-1}} \circ I"] & \\
\mathcal{C} \arrow[rr, "F_{(\Sigma^{\perp\top})^{-1}}"'] & & \mathcal{C}[(\Sigma^{\perp\top})^{-1}]
\end{tikzcd}$$

The following statements are equivalent:

(i) I has a left adjoint R;

(ii) $F_{(\Sigma^{\perp\top})^{-1}}$ has a fully faithful right adjoint;

(iii) $F_{(\Sigma^{\perp\top})^{-1}} \circ I$ is essentially surjective on objects;

(iv) $F_{(\Sigma^{\perp\top})^{-1}} \circ I$ is an equivalence.

In this case, we have

$$\{f \in \mathcal{C} \mid F_{(\Sigma^{\perp\top})^{-1}}(f) \text{ is an isomorphism}\} = \{f \in \mathcal{C} \mid R(f) \text{ is an isomorphism}\} = \Sigma^{\perp\top}$$

and this class of morphisms admits a left calculus of fractions.

Proof. It suffices to use Theorem 5.3 with $\Sigma^{\perp\top}$ instead of Σ since $\Sigma^{\perp\top\perp} = \Sigma^{\perp}$. Notice that when I has a left adjoint, the universal property of the unit η exactly means that $\eta_C \in \Sigma^{\perp\top}$ for all objects C in $\mathcal{C}$. ■

6. An application: the Injective Subcategory Problem

As an application of Theorem 4.1, we prove in this final section the weak version of Theorem 5.3. The weak version of the OSP is the Injective Subcategory Problem (ISP) and deals with the question to know whether $\Sigma^{\triangle}$ is weakly reflective in $\mathcal{C}$. As we are going to show, this is linked with the existence of some sort of weak adjunction involving $F_{1/\Sigma} \colon \mathcal{C} \to \mathcal{C}[1/\Sigma]$.

6.1. Theorem. *Let Σ be a set of morphisms in a category $\mathcal{C}$. Denote by $J \colon \Sigma^{\triangle} \hookrightarrow \mathcal{C}$ the full inclusion functor.*

$$\begin{tikzcd} & \Sigma^{\triangle} \arrow[dl, hook', "J"'] \arrow[dr, "F_{1/\Sigma}\circ J"] & \\ \mathcal{C} \arrow[rr, "F_{1/\Sigma}"'] & & \mathcal{C}[1/\Sigma] \end{tikzcd}$$

The following statements are equivalent:

(i) each object $C \in \mathcal{C}$ has a weak reflection $n_C \colon C \to J(D)$ along J with $n_C \in \Sigma$;

(ii) there exists

- *a lax-*Pos*-functor $G \colon \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \rightharpoonup \mathcal{P}_0(\mathcal{C})$,*
- *a natural* Pos*-transformation $\eta \colon 1_{\mathcal{P}_0(\mathcal{C})} \to G \circ \mathcal{P}_0(F_{1/\Sigma})$ and*
- *a* Pos*-transformation $\varepsilon \colon \mathcal{P}_0(F_{1/\Sigma}) \circ G \rightsquigarrow 1_{\mathcal{P}_0(\mathcal{C}[1/\Sigma])}$*

satisfying

- *for all object $C \in \mathcal{C}$, $\eta_C = \{n_C\}$ for some $n_C \in \Sigma$,*

- $1_{G(D)} \leqslant G(\varepsilon_D) \circ \eta_{G(D)}$ *for all object* D *in* $\mathcal{C}[1/\Sigma]$;

(iii) there exist G, η *and* ε *as in (ii) and satisfying, in addition to the above,*

- G *is almost complete,*
- $\varepsilon_{\mathcal{P}_0(F_{1/\Sigma})(C)} \circ \mathcal{P}_0(F_{1/\Sigma})(\eta_C) = 1_{\mathcal{P}_0(F_{1/\Sigma})(C)}$ *for any object* C *in* $\mathcal{C}$.

In this case, we have

$$\Sigma^{\vartriangle\triangledown} = \{f \in \mathcal{C} \mid F_{1/\Sigma}(f) \text{ is a split monomorphism}\}$$

and this class of morphisms contains all split monomorphisms, is closed under composition and given any span $f\colon A \to B$, $g\colon A \to C$ *in* $\mathcal{C}$ *with* $f \in \Sigma^{\vartriangle\triangledown}$,

$$\begin{tikzcd} A \arrow[r, "f\in\Sigma^{\vartriangle\triangledown}"] \arrow[d, "g"'] & B \arrow[d, dotted, "g'"] \\ C \arrow[r, dotted, "f'\in\Sigma^{\vartriangle\triangledown}"'] & D \end{tikzcd}$$

one can find f' *and* g' *in* $\mathcal{C}$ *such that* $f' \in \Sigma^{\vartriangle\triangledown}$ *and* $f'g = g'f$.

Proof. Suppose (i). By Theorem 3.6, we know $\mathcal{P}_0(J)$ has an almost complete weak left adjoint $R\colon \mathcal{P}_0(\mathcal{C}) \rightharpoonup \mathcal{P}_0(\Sigma^{\vartriangle})$ given by

$$R(f) = \{g\colon R(A) \to R(B) \mid J(g)n_A = n_B f\}$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "n_A"'] & B \arrow[d, "n_B"] \\ JR(A) \arrow[r, dotted, "J(g)"'] & JR(B) \end{tikzcd}$$

for any $f\colon A \to B$ in $\mathcal{C}$. It comes equipped with a natural Pos-transformation $\eta\colon 1_{\mathcal{P}_0(\mathcal{C})} \to \mathcal{P}_0(J) \circ R$ given by $\eta_C = \{n_C\}$ for $C \in \mathcal{C}$ and a Pos-transformation $\theta\colon R \circ \mathcal{P}_0(J) \rightsquigarrow 1_{\mathcal{P}_0(\Sigma^{\vartriangle})}$ such that $\mathcal{P}_0(J)(\theta_K) \circ \eta_{\mathcal{P}_0(J)(K)} = 1_{\mathcal{P}_0(J)(K)}$ for each $K \in \Sigma^{\vartriangle}$. Now, let us check R satisfies conditions (A) and (B) of Theorem 4.1. Given any $s\colon A \to B$ in Σ,

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "n_A"'] & B \arrow[d, "n_B"] \arrow[dl, dotted, "k"'] \\ JR(A) & JR(B) \arrow[l, dotted, "d"] \end{tikzcd}$$

since $R(A) \in \Sigma^{\vartriangle}$, there exists an $k\colon B \to JR(A)$ such that $ks = n_A$. Since $(R(B), n_B)$ is a weak reflection, there exists $d\colon R(B) \to R(A)$ such that $dn_B = k$. Given $f\colon C \to A$ in $\mathcal{C}$ and $g \in R(sf)$, then $dg \in R(f)$ since

$$dgn_C = dn_B sf = ksf = n_A f$$

proving condition (A). Now, given $f, f' \colon C \to A$ such that $R(sf) = R(sf')$, $h \colon A \to D$ and $l \in R(hf)$, we need to prove $l \in R(hf')$.

$$\begin{tikzcd}
C \arrow[r, shift left, "f"] \arrow[r, shift right, "f'"'] \arrow[dd, "n_C"'] & A \arrow[rr, "s"] \arrow[dr, "h"] \arrow[dd, "n_A"'] & & B \arrow[dd, "n_B"] \\
& & D \arrow[dd, "n_D"] & \\
JR(C) \arrow[rrr, dotted, bend left, "g"] \arrow[dr, bend right, "l"'] & JR(A) \arrow[dr, dotted, "m"] & & JR(B) \arrow[ll, "d"] \\
& & JR(D) &
\end{tikzcd}$$

Pick any $g \in R(sf) = R(sf')$ and any $m \in R(h)$. By condition (A), we already know that $dg \in R(f)$. We then have $l \in R(hf')$ since

$$ln_C = n_D hf = mn_A f = mdgn_C = mdn_B sf' = mn_A f' = n_D hf'$$

proving condition (B). Therefore, by Theorem 4.1, there exists an almost complete lax-Pos-functor $R' \colon \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \rightharpoonup \mathcal{P}_0(\Sigma^\triangle)$ such that $R' \circ \mathcal{P}_0(F_{1/\Sigma}) = R$.

$$\begin{tikzcd}
& \mathcal{P}_0(\Sigma^\triangle) \arrow[dl, hook, shift right, "\mathcal{P}_0(J)"'] \arrow[dr, shift left, "\mathcal{P}_0(F_{1/\Sigma}) \circ \mathcal{P}_0(J)"] & \\
\mathcal{P}_0(\mathcal{C}) \arrow[ur, shift right, "R"'] \arrow[rr, "\mathcal{P}_0(F_{1/\Sigma})"'] & & \mathcal{P}_0(\mathcal{C}[1/\Sigma]) \arrow[ul, shift left, "R'"]
\end{tikzcd}$$

Let $G = \mathcal{P}_0(J) \circ R'$. Since $G \circ \mathcal{P}_0(F_{1/\Sigma}) = \mathcal{P}_0(J) \circ R' \circ \mathcal{P}_0(F_{1/\Sigma}) = \mathcal{P}_0(J) \circ R$, the natural Pos-transformation $\eta \colon 1_{\mathcal{P}_0(\mathcal{C})} \to \mathcal{P}_0(J) \circ R$ is the required one. Now, given any object $D \in \mathcal{C}[1/\Sigma]$, we define

$$\varepsilon_D = \{e \in \mathcal{C}[1/\Sigma](JR(D), D) \mid e \circ [n_D] = 1_D\}.$$

This set is non-empty since $[\overline{n_D}] \in \varepsilon_D$. Hence, this defines a Pos-transformation $\varepsilon \colon \mathcal{P}_0(F_{1/\Sigma}) \circ G \rightsquigarrow 1_{\mathcal{P}_0(\mathcal{C}[1/\Sigma])}$ satisfying $\varepsilon_{\mathcal{P}_0(F_{1/\Sigma})(C)} \circ \mathcal{P}_0(F_{1/\Sigma})(\eta_C) = 1_{\mathcal{P}_0(F_{1/\Sigma})(C)}$ for each object C in $\mathcal{C}$. To prove the inequality in (ii), we consider, for any object $D \in \mathcal{C}[1/\Sigma]$, a morphism $d \colon RJR(D) \to R(D)$ such that $dn_{JR(D)} = 1_{JR(D)}$.

$$\begin{tikzcd}
JR(D) \arrow[r, "n_{JR(D)}"] \arrow[d, "1_{JR(D)}"'] & JRJR(D) \arrow[dl, dotted, "d"] \\
JR(D) &
\end{tikzcd}$$

In particular, $dn_{JR(D)}n_D = n_D$, proving that $\{d\} \circ R(n_D f) \subseteq R(f)$ for any $f \in \mathcal{C}(D', D)$ by the above argument. By the construction of R' given in Theorem 4.1, this immediately shows that $d \in R'([\overline{n_D}])$. Therefore,

$$\begin{aligned} G(\varepsilon_D) \circ \eta_{G(D)} &= \mathcal{P}_0(J)(R'(\varepsilon_D)) \circ \{n_{JR(D)}\} \\ &\supseteq \mathcal{P}_0(J)(R'([\overline{n_D}])) \circ \{n_{JR(D)}\} \\ &\ni d \circ n_{JR(D)} \\ &= 1_{JR(D)} \\ &= 1_{G(D)} \end{aligned}$$

proving (iii).

(iii) $\Rightarrow$ (ii) being trivial, let us prove (ii) $\Rightarrow$ (i). Let us first prove that if C is an object of $\mathcal{C}[1/\Sigma]$ (i.e., of $\mathcal{C}$), then $G(C) \in \Sigma^{\vartriangle}$. Let $s\colon A \to B$ in Σ and $f\colon A \to G(C)$ in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[rr, "s"] \arrow[dr, "n_A"'] \arrow[dd, "f"'] & & B \arrow[dr, "n_B"] & \\ & G(A) \arrow[rr, Rightarrow, shift left, "G(\mathcal{P}_0(F_{1/\Sigma})(s))"] \arrow[dd, Rightarrow, "G(\mathcal{P}_0(F_{1/\Sigma})(f))"] & & G(B) \arrow[ll, Rightarrow, shift left, "G([\overline{s}])"] \\ G(C) & & & \\ & G(\mathcal{P}_0(F_{1/\Sigma})(G(C))) \arrow[ul, Rightarrow, "G(\varepsilon_C)"] & & \end{tikzcd}$$

We know that

$$\begin{aligned} G([\overline{s}]) \circ G(\mathcal{P}_0(F_{1/\Sigma})(s)) \circ \eta_A &\subseteq G([\overline{s}] \circ \mathcal{P}_0(F_{1/\Sigma})(s)) \circ \eta_A \\ &= G(1_A) \circ \eta_A \\ &= G(\mathcal{P}_0(F_{1/\Sigma})(1_A)) \circ \eta_A \\ &= \eta_A \circ 1_A \\ &= \eta_A. \end{aligned}$$

Since η_A is the singleton $\{n_A\}$, this proves that $G([\overline{s}]) \circ G(\mathcal{P}_0(F_{1/\Sigma})(s)) \circ \eta_A = \eta_A$. We can thus compute

$$\begin{aligned} & G(\varepsilon_C) \circ G(\mathcal{P}_0(F_{1/\Sigma})(f)) \circ G([\overline{s}]) \circ \eta_B \circ \{s\} \\ &= G(\varepsilon_C) \circ G(\mathcal{P}_0(F_{1/\Sigma})(f)) \circ G([\overline{s}]) \circ G(\mathcal{P}_0(F_{1/\Sigma})(s)) \circ \eta_A \\ &= G(\varepsilon_C) \circ G(\mathcal{P}_0(F_{1/\Sigma})(f)) \circ \eta_A \\ &= G(\varepsilon_C) \circ \eta_{G(C)} \circ \{f\} \\ &\ni f \end{aligned}$$

where the last line is deduced from $1_{G(C)} \in G(\varepsilon_C) \circ \eta_{G(C)}$. Therefore, there must exist $g \in G(\varepsilon_C) \circ G(\mathcal{P}_0(F_{1/\Sigma})(f)) \circ G([\overline{s}]) \circ \eta_B$ such that $gs = f$, proving that $G(C) \in \Sigma^{\vartriangle}$. Now, $n_C \colon C \to G(C) = JG(C)$ is a weak reflection for C along J since $n_C \in \Sigma$. We have thus proved (i).

Now, let us suppose these conditions hold. We use the notation of the proof of (i) $\Rightarrow$ (iii). For any $f \colon A \to B \in \Sigma^{\vartriangle\triangledown}$, there exists $k \colon B \to JR(A)$ such that $kf = n_A$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "n_A"'] & B \arrow[dl, dotted, "k"] \\ JR(A) & \end{tikzcd}$$

We thus have

$$[\overline{n_A}] \circ F_{1/\Sigma}(k) \circ F_{1/\Sigma}(f) = [\overline{n_A}] \circ F_{1/\Sigma}(n_A) = 1_A$$

in $\mathcal{C}[1/\Sigma]$, proving that $F_{1/\Sigma}(f)$ is a split monomorphism. Conversely, if $f \in \mathcal{C}(A, B)$ is such that $F_{1/\Sigma}(f)$ is a split monomorphism, choose $[p] \in \mathcal{C}[1/\Sigma](B, A)$ such that $[p] \circ F_{1/\Sigma}(f) = 1_A$. Given any $x \in R'([p])$ and any $g \in R(f)$, we have

$$xg \in x \circ R(f) \subseteq R'([p]) \circ R'(F_{1/\Sigma}(f)) \subseteq R'([p] \circ F_{1/\Sigma}(f)) = R'(1_A) = R(1_A).$$

This means $xgn_A = n_A$. Now, given any $k \in \mathcal{C}(A, J(K))$ with $K \in \Sigma^{\vartriangle}$, we know there exists $h \colon R(A) \to K$ such that $hn_A = k$.

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[dl, "k"'] \arrow[d, "n_A"] & B \arrow[d, "n_B"] \\ J(K) & JR(A) \arrow[l, dotted, "h"] \arrow[r, shift left, "g"] & JR(B) \arrow[l, shift left, "x"] \end{tikzcd}$$

Then, we have

$$hxn_Bf = hxgn_A = hn_A = k$$

proving that $f \in \Sigma^{\vartriangle\triangledown}$. Hence we have

$$\Sigma^{\vartriangle\triangledown} = \{f \in \mathcal{C} \mid F_{1/\Sigma}(f) \text{ is a split monomorphism}\}$$

and this class of morphisms obviously contains split monomorphisms and is closed under composition.

Finally, let $f \colon A \to B$ and $g \colon A \to C$ in $\mathcal{C}$ with $f \in \Sigma^{\vartriangle\triangledown}$. Since $R(C) \in \Sigma^{\vartriangle}$, this implies that there exists $g' \colon B \to JR(C)$ making the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, dotted, "g'"] \\ C \arrow[r, "n_C"'] & JR(C) \end{tikzcd}$$

commutative. This proves the required condition since $n_C \in \Sigma \subseteq \Sigma^{\vartriangle\triangledown}$. ∎

References

[1] J. Adámek, M. Hébert and L. Sousa, The orthogonal subcategory problem and the small object argument, *Appl. Categor. Struct.* **17** (2009), 211–246.

[2] J. Adámek, H. Herrlich, J. Rosický and W. Tholen, On a generalized small-object argument for the injective subcategory problem, *Cah. Topol. Géom. Différ. Catég.* **43** (2002), 83–106.

[3] J. Adámek, J. Rosický and V. Trnková, Are all limit-closed subcategories of locally presentable categories reflective?, in *Categorical Algebra and its Applications*, Lecture Notes in Mathematics **1348**, Springer (1988), 1–18.

[4] F. Borceux, Handbook of Categorical Algebra I, *Encyclopedia of Mathematics and its Applications* **50**, Cambridge University Press (1994).

[5] A. Carboni and E.M. Vitale, Regular and exact completions, *J. Pure Appl. Algebra* **125** (1998), 79–116.

[6] C. Casacuberta and B. Chorny, The orthogonal subcategory problem in homotopy theory, in *An Alpine anthology of homotopy theory*, Contemporary Mathematics **399**, American Mathematical Society (2006), 41–54.

[7] C. Casacuberta and A. Frei, On saturated classes of morphisms, *Theory Appl. Categ.* **7** (2000), 43–46.

[8] A. Frei, On completion and shape, *Bol. Soc. Brasil. Mat.* **5** (1974), 147–159.

[9] P.J. Freyd, Representations in abelian categories, in *Proceedings of the Conference on Categorical Algebra*, Springer (1966), 95–120.

[10] P.J. Freyd and G.M. Kelly, Categories of continuous functors, I, *J. Pure Appl. Algebra* **2** (1972), 169–191.

[11] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **35**, Springer (1967).

[12] W. Gähler, Galois connections in category theory, topology and logic, *Galois Connections and Applications*, Springer (2004), 425–452.

[13] R. Garner, Understanding the small object argument, *Appl. Categor. Struct.* **17** (2009), 247–285.

[14] M. Hébert, λ-presentable morphisms, injectivity and (weak) factorization systems, *Appl. Categor. Struct.* **14** (2006), 273–289.

[15] S.N. Hosseini, A.R. Shir Ali Nasab and W. Tholen, Fraction, restriction and range categories from non-monic classes of morphisms, *arXiv:1903.00081v2*.

[16] P.T. Johnstone, Sketches of an elephant: a topos theory compendium. Vol. 1. *Oxford Logic Guides* **43**, Oxford University Press (2002).

[17] P.C. Kainen, Weak adjoint functors, *Math. Z.* **122** (1971), 1–9.

[18] G.M. Kelly, A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on, *Bull. Austral. Math. Soc.* **22** (1980), 1–83.

[19] S. Lack and J. Rosický, Enriched weakness, *J. Pure Appl. Algebra* **216** (2012), 1807–1822.

[20] S. Mac Lane and I. Moerdijk, Sheaves in geometry and logic, A first introduction to topos theory, *Universitext*, Springer (1994).

[21] L. Sousa, A calculus of lax fractions, *J. Pure Appl. Algebra* **221** (2017), 422–448.

Department of Mathematics and Statistics
University of Ottawa
150 Louis-Pasteur
K1N 6N5
Ottawa, Ontario
Canada

and

Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
Chemin du cyclotron 2, bte L7.01.02
B–1348
Louvain-la-Neuve
Belgium

Email: pjacqmin@uottawa.ca, pierre-alain.jacqmin@uclouvain.be

This article may be accessed at http://www.tac.mta.ca/tac/

THEORY AND APPLICATIONS OF CATEGORIES will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

SUBSCRIPTION INFORMATION Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. Full text of the journal is freely available at `http://www.tac.mta.ca/tac/`.

INFORMATION FOR AUTHORS LaTeX2e is required. Articles may be submitted in PDF by email directly to a Transmitting Editor following the author instructions at
`http://www.tac.mta.ca/tac/authinfo.html`.